LECTURE 1.5

Solving Models with Rational Expectations

ECON42240 Advanced Macroeconomics

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Where We Are

Last Week's Lecture

Last week extracted a variable nobody observes, from the data depending on it:

1. **Unobserved Components.**
   Writing a model in state-space form.
2. **The Kalman Filter.**
   Extracting a state you cannot see.
3. **Estimation and Smoothing.**
   Estimating the variances, and revising the past.
4. **The Natural Rate of Interest.**
   What the filter is used for.

Lecture Plan

1. **Rational Expectations.**
   The assumption, and the forward solution it delivers.
2. **Reduced Forms and Persistence.**
   A solved model as a VAR, and lagged dynamics.
3. **Solving the System.**
   Eigenvalues, the Blanchard–Kahn count, and generalised Schur.
4. **The Lucas Critique.**
   Why reduced-form coefficients shift when policy changes.

SECTION 1

Rational Expectations

What a VAR Omits About Behaviour

A VAR is a dynamic stochastic model with no behaviour behind its coefficients:

- **No Decision Rules:** No choice by a firm or household appears in a VAR.
- **No Reason:** A VAR records that a response happened, not why.
- **The Cost:** Forecasts and policy analysis need a mechanism to be believed.

A DSGE Model Has the Mechanism a VAR Lacks

Dynamic stochastic general equilibrium models are built to match the VAR evidence. Optimising firms and households generate the responses.

Forward-Looking Behaviour Needs an Expectation

Only forward dynamics need an assumption about how people form expectations:

- **Backward Dynamics:** An identity carries last period's capital into today's.
- **Forward Dynamics:** Investment pays off later, so it depends on what firms expect.

What Rational Expectations Assumes

Before the 1970s, macroeconomists generally assumed people extrapolated from recent data. Lucas and Sargent promoted a stronger assumption:

- **Efficient Use of Information:** People use public data, making no systematic errors.
- **Knowledge of the Model:** People understand the model's structure and forecast from it.

> **Rational Expectations Is Model-Consistent Forecasting**
> Agents inside the model forecast using the model itself. Their forecasts and the model's predictions agree.

Rational Expectations as a Baseline

Rational expectations is a strong assumption, because nobody knows the economy's structure. The argument for it as a starting point is narrow:

- **The Alternative Is Systematic Error:** Any other rule leaves people predictably wrong.
- **No Agreed Replacement:** The alternatives are many, and none is agreed at present.
- **The Test Is the Data:** Rational expectations models are judged on their fit.

> **Behavioural Evidence Against Rationality Is Extensive**
> Behavioural economists have documented many departures from rationality. Whelan's defence is comparative, not absolute.

What Would You Put in Its Place?

Rational expectations gives everyone in the model the modeller's own knowledge of it. Nobody believes that literally.

> **Question**
> Suppose you drop rational expectations from a model of inflation. What do you write down instead, and what does the choice cost?

Take answers from the room.

The First-Order Stochastic Difference Equation

Many models set today's value from a driving variable and tomorrow's expected value:

$$y_t = x_t + \alpha \, E_t y_{t+1} \tag{1}$$

where:

- y_t: endogenous variable
- x_t: driving variable
- α: forward weight
- E_t: time-t expectation

> **A Stochastic Difference Equation Links Dates**
> A difference equation links a variable's values at different dates. A stochastic one adds random shocks.

Tomorrow's Equation, Forecast Today

The equation holds every period, so it also holds tomorrow:

$$y_{t+1} = x_{t+1} + a\, E_{t+1}\, y_{t+2} \tag{2}$$

Take today's expectation of both sides:

$$E_t\, y_{t+1} = E_t\, x_{t+1} + a\, E_t\, E_{t+1}\, y_{t+2} \tag{3}$$

The Law of Iterated Expectations

The law of iterated expectations collapses the doubled forecast to today's:

$$E_t y_{t+1} = E_t x_{t+1} + \alpha\, E_t y_{t+2} \tag{4}$$

Today's Forecast of Tomorrow's Forecast Is Today's Forecast
Nobody rationally expects to hold a different forecast next period. The law of iterated expectations is the standard name.

Substituting Forward Swaps an Expectation for Two Terms

The marked term is the left side of tomorrow's equation. Put one inside the other:

$$y_t = x_t + \alpha \left(E_t x_{t+1} + \alpha E_t y_{t+2} \right) \tag{5}$$

Multiplying Out Leaves Two Driving Terms

Multiply the bracket out by a. Two terms in x and one in y remain:

$$y_t = x_t + a\, E_t x_{t+1} + a^2\, E_t y_{t+2} \tag{6}$$

The model dated $t + 2$, forecast today, gives the last term the same way:

$$E_t y_{t+2} = E_t x_{t+2} + a\, E_t y_{t+3} \tag{7}$$

A Second Substitution Adds a Third Driving Term

Put that line in place of $E_t y_{t+2}$:

$$y_t = x_t + a\, E_t x_{t+1} + a^2 \left(E_t x_{t+2} + a\, E_t y_{t+3} \right) \tag{8}$$

Multiply the bracket out by a^2:

$$y_t = x_t + a\, E_t x_{t+1} + a^2\, E_t x_{t+2} + a^3\, E_t y_{t+3} \tag{9}$$

Repeating the Substitution to the Limit

Each substitution adds a term in x, one power of a higher. After N of them:

$$y_t = \sum_{k=0}^{N-1} a^k E_t x_{t+k} \; + \; a^N E_t y_{t+N} \tag{10}$$

where:

- N: number of substitutions
- k: periods ahead

> **The Exponent Counts Periods Ahead**
> Expected x at k periods ahead carries the weight a^k. With $|a| < 1$, distant news counts for less.

The Terminal Condition on the Remaining Term

Assume the remaining term vanishes as the horizon lengthens:

$$\lim_{N\to\infty} a^N E_t y_{t+N} = 0 \qquad \Longrightarrow \qquad \underbrace{y_t = \sum_{k=0}^{\infty} a^k E_t x_{t+k}}_{\text{the discounted forward sum}} \tag{11}$$

The Terminal Condition Is Assumed, Not Derived

The limit rules out paths on which y grows as fast as a^{-1}. Dropping it admits rational bubbles.

Pricing an Asset with No Risk Premium

A risk-neutral investor holds an asset only while it earns the safe return:

$$\frac{\overbrace{D_t + E_t P_{t+1}}^{\text{next period's payoff}}}{P_t} = 1 + r \tag{12}$$

where:

- P_t: asset price
- D_t: dividend
- r: safe return

Rearranging the Condition for Today's Price

Multiply both sides by P_t, which clears the fraction:

$$D_t + E_t P_{t+1} = (1 + r)\, P_t \tag{13}$$

Divide both sides by $1 + r$, and read right to left:

$$P_t = \frac{D_t}{1 + r} + \frac{1}{1 + r} E_t P_{t+1} \tag{14}$$

The Price Equation Is the First-Order Equation

Set the price equation under the first-order equation, and mark the matching pieces:

$$
\begin{aligned}
\boxed{y_t} &= \boxed{x_t} + \boxed{a}\, E_t\, \boxed{y_{t+1}} \\
\boxed{P_t} &= \boxed{\frac{D_t}{1+r}} + \boxed{\frac{1}{1+r}}\, E_t\, \boxed{P_{t+1}}
\end{aligned}
\tag{15}
$$

The matches, piece by piece:

$$
\boxed{y = P}, \qquad \boxed{x_t = \frac{D_t}{1+r}}, \qquad \boxed{a = \frac{1}{1+r}} \tag{16}
$$

A Price Is a Discounted Sum of Expected Dividends

With $r > 0$, the weight is below one. The forward solution with the matches dropped in:

$$P_t = \sum_{k=0}^{\infty} \left(\frac{1}{1+r} \right)^k E_t \frac{D_{t+k}}{1+r} \tag{17}$$

The dividend's own discount joins the others, so the power rises by one:

$$P_t = \sum_{k=0}^{\infty} \left(\frac{1}{1+r} \right)^{k+1} E_t D_{t+k} \tag{18}$$

The Same Equation Solved Backwards

Swap the expectation for outcome plus unpredictable forecast error (ϵ_t), then shift back:

$$y_t = a^{-1}y_{t-1} - a^{-1}x_{t-1} - \epsilon_t \tag{19}$$

Substituting the rule into itself, again and again, leaves two backward sums:

$$y_t = -\sum_{k=0}^{\infty} a^{-k}\epsilon_{t-k} - \sum_{k=1}^{\infty} a^{-k}x_{t-k} \tag{20}$$

The Size of $|a|$ Decides Which Solution to Use

Both solutions satisfy the equation, and so does any mix of them. Which weights explode depends on $|a|$:

- **Weight Below One:** With $|a| < 1$ the backward weights explode, so solve forward.
- **Weight Above One:** With $|a| > 1$ the forward weights explode, so solve backwards.

> **Distant News Should Not Dominate Today's Value**
> With $|a| > 1$ today's y depends more on the far future than the present. Whelan calls that dependence uncomfortable, even where the sum converges.

The Path of x Pins Down y Only When $|a| < 1$

With $|a| < 1$ the path of x delivers one path of y. With $|a| > 1$ the equation also holds for any unpredictable error.

> **Determinacy Is Uniqueness of the Stable Solution**
> A model is determinate when exactly one non-explosive path satisfies it. Indeterminacy means many do.

An indeterminate model cannot predict y, even from the full path of x.

What Else Is Needed to Simulate y?

The forward solution writes y_t as a sum of expectations of future x.

> **Question**
> You want to simulate y on a computer. What must you add to the model before you can?

Take answers from the room.

SECTION 2

Reduced Forms and Persistence

From Structural Equation to Reduced Form

An AR(1) for x, hit by fresh shocks (ϵ_t), makes every expectation computable:

$$x_t = \rho x_{t-1} + \epsilon_t, \quad |\rho| < 1 \qquad \Longrightarrow \qquad E_t x_{t+k} = \rho^k x_t \tag{21}$$

where:

- ρ: persistence of x

Lecture 1.1 built the AR(1).

A Forward-Looking Model Generates VAR-Shaped Data

With $|a\rho| < 1$ the forward sum collapses to one coefficient on x_t:

$$y_t = \frac{1}{1 - a\rho} x_t \tag{22}$$

With last period's y in place of x, the model looks only backwards:

$$y_t = \rho\, y_{t-1} + \frac{1}{1 - a\rho} \epsilon_t \tag{23}$$

Every DSGE Model Is Built the Same Way

Every DSGE model in the module's back half is built in three steps:

1. **Structural Equations:** Derive equations containing expected future driving variables.
2. **Driving Processes:** Assume a time series process for each driving variable.
3. **Reduced Form:** Solve for a representation a computer can simulate.

> **Only the Third Step Needs a Solution Method**
> Steps one and two are modelling choices. The solution methods in this lecture are step three.

Why the Root Is ρ Alone

In the reduced form, y inherits its autoregressive root ρ from x.

Question

The forward-looking weight a vanished from the root and survives in the coefficient. Why should a have no effect at all on persistence?

Take answers from the room.

Jump Variables and the Real Economy

With no lagged y, all persistence in y comes from x.

> **A Jump Variable Resets When Expectations Change**
> A jump variable depends only on today and the expected future. Nothing in the past restricts its value.

The description suits stock prices better than employment, consumption or investment.

The Second-Order Stochastic Difference Equation

Many macroeconomic models add the past to the right-hand side. The past carries the backward weight (a), a new job for that letter:

$$y_t = a\, y_{t-1} + b\, E_t y_{t+1} + x_t \tag{24}$$

where:

- b: forward weight

> **Second-Order Means a Span of Two Periods**
> The equation spans $t-1$ to $t+1$, which is three dates. The order is named by the span, not by the count.

A Change of Variable That Removes the Lag

Suppose some λ makes v_t obey a first-order equation. The forward solution then applies:

$$v_t \equiv y_t - \lambda y_{t-1}, \qquad v_t = \alpha\, E_t v_{t+1} + \beta x_t \tag{25}$$

where:

- v_t: constructed variable
- λ: unknown coefficient

> **Both Weights Are Placeholders for Now**
> The placeholder weights (α, β) come out of the derivation. This v_t is not Lecture 1.4's measurement error.

Adding the Lag Back Gives Today's and Tomorrow's y

Add λy_{t-1} to both sides of the definition, and read right to left:

$$y_t = \boxed{v_t + \lambda y_{t-1}} \tag{26}$$

The definition holds every period, so one period on it gives tomorrow's y:

$$y_{t+1} = v_{t+1} + \lambda y_t \tag{27}$$

Substituting Leaves One y_t Still to Replace

Forecast tomorrow's line today. Today's y_t is already known:

$$E_t y_{t+1} = E_t v_{t+1} + \lambda y_t \tag{28}$$

Put both lines into the model, in place of y_t and $E_t y_{t+1}$:

$$v_t + \lambda y_{t-1} = a\, y_{t-1} + b \left(E_t v_{t+1} + \lambda y_t\right) + x_t \tag{29}$$

Replacing the Last y_t Brings in Another Lag

Multiply the bracket out by b. One y_t remains on the right:

$$v_t + \lambda y_{t-1} = a\, y_{t-1} + b\, E_t v_{t+1} + b\lambda\, y_t + x_t \tag{30}$$

Replace it by the definition, $v_t + \lambda y_{t-1}$, once more:

$$v_t + \lambda y_{t-1} = a\, y_{t-1} + b\, E_t v_{t+1} + b\lambda \left(v_t + \lambda y_{t-1}\right) + x_t \tag{31}$$

Moving Terms Leaves v_t on the Left, Lags on the Right

Multiply the new bracket out by $b\lambda$:

$$v_t + \lambda y_{t-1} = a\, y_{t-1} + b\, E_t v_{t+1} + b\lambda\, v_t + b\lambda^2 y_{t-1} + x_t \tag{32}$$

Subtract $b\lambda\, v_t$ and λy_{t-1} from both sides:

$$v_t - b\lambda\, v_t = b\, E_t v_{t+1} + x_t + b\lambda^2 y_{t-1} - \lambda y_{t-1} + a\, y_{t-1} \tag{33}$$

Collecting Terms Gives the Characteristic Equation

Factor out v_t on the left and y_{t-1} on the right:

$$(1 - b\lambda)\, v_t = b\, E_t v_{t+1} + x_t + \left(b\lambda^2 - \lambda + a\right) y_{t-1} \tag{34}$$

The Bracket Must Be Zero

Setting $b\lambda^2 - \lambda + a = 0$, the characteristic equation, makes v_t first-order. The quadratic has two roots.

With the Bracket at Zero, v_t Is First-Order

With λ a root, the bracket is zero, so the lagged term drops:

$$(1 - b\lambda)\, v_t = b\, E_t v_{t+1} + x_t \tag{35}$$

Divide both sides by $1 - b\lambda$:

$$v_t = \frac{b}{1 - b\lambda} E_t v_{t+1} + \frac{1}{1 - b\lambda} x_t \tag{36}$$

The Placeholder Weights Take Their Values

Set v_t's equation under the placeholder form, and mark the matching pieces:

$$\begin{aligned} v_t &= \boxed{\alpha} \; E_t v_{t+1} + \boxed{\beta} \; x_t \\ v_t &= \boxed{\frac{b}{1-b\lambda}} E_t v_{t+1} + \boxed{\frac{1}{1-b\lambda}} x_t \end{aligned} \tag{37}$$

The two matches:

$$\boxed{\alpha = \frac{b}{1-b\lambda}}, \qquad \boxed{\beta = \frac{1}{1-b\lambda}} \tag{38}$$

The Other Root Sets the Forward Discount Factor

The characteristic roots (λ_1, λ_2) solve one quadratic. Each pins down the other:

$$\underbrace{\frac{b}{1 - b\lambda_1}}_{\text{forward discount factor}} = \frac{1}{\lambda_2} \tag{39}$$

The Sum of the Roots Links Them

The roots add to one over b, so $1 - b\lambda_1 = b\lambda_2$. Dividing b by each side gives the identity.

Each Part of the Solution Constrains One Root

The solution has a backward part and a forward part, one root each:

- **The Backward Part:** $y_t = \lambda_1 y_{t-1} + v_t$ is non-explosive only if $|\lambda_1| \leq 1$.
- **The Forward Part:** The sum discounts at $1/\lambda_2$, converging only if $|\lambda_2| > 1$.

Lecture 1.1's lag-polynomial roots are these roots inverted.

Two Conditions Reduce to One
One root must lie outside the unit circle. The other lies on or inside it.

The Forward Solution Applied to the Constructed Variable

With $\lambda = \lambda_1$, set v_t's equation under the first-order equation, and mark the matches:

$$\begin{aligned} y_t &= x_t + a\, E_t\, y_{t+1} \\ v_t &= \beta x_t + \alpha\, E_t\, v_{t+1} \end{aligned} \tag{40}$$

The forward solution with those matches, then the constant β taken out:

$$v_t = \sum_{k=0}^{\infty} \alpha^k E_t\, \beta x_{t+k} = \beta \sum_{k=0}^{\infty} \alpha^k E_t\, x_{t+k} \tag{41}$$

The Solution to the Second-Order Equation

Put in the two weights' values:

$$v_t = \frac{1}{1-b\lambda} \sum_{k=0}^{\infty} \left(\frac{b}{1-b\lambda} \right)^k E_t x_{t+k} \tag{42}$$

Add λy_{t-1} to both sides, since $y_t = v_t + \lambda y_{t-1}$:

$$y_t = \lambda y_{t-1} + \frac{1}{1-b\lambda} \sum_{k=0}^{\infty} \left(\frac{b}{1-b\lambda} \right)^k E_t x_{t+k} \tag{43}$$

Persistence Now Has Two Sources

The lagged term carries the model's own dynamics. The sum carries expected x.

A Hybrid New Keynesian Phillips Curve

Many studies find the forward-looking New Keynesian Phillips curve misses inflation persistence. A hybrid version adds lagged inflation:

$$\pi_t = \gamma_f E_t \pi_{t+1} + \gamma_b \pi_{t-1} + \kappa x_t \tag{44}$$

where:

- π_t: inflation rate
- γ_f: forward weight
- γ_b: backward weight
- κ: slope on inflationary pressure

> **Lecture 2.2 Teaches the Phillips Curve Itself**
> Here the curve is a worked example of the method. Its economics and evidence come in Lecture 2.2.

The Hybrid Curve Is a Second-Order Equation

Let the forward weight (θ) and backward weight sum to one. Set the curve under the general form:

$$\begin{aligned} y_t &= a \; y_{t-1} + b \, E_t y_{t+1} + x_t \\ \pi_t &= (1-\theta) \, \pi_{t-1} + \theta \, E_t \pi_{t+1} + \kappa x_t \end{aligned} \tag{45}$$

So $b = \theta$ and $a = 1 - \theta$ go into $b\lambda^2 - \lambda + a = 0$:

$$\theta \, \lambda^2 - \lambda + (1-\theta) = 0 \tag{46}$$

The Weight θ Decides Which Root the Hybrid Curve Uses

The characteristic equation's two roots are:

$$\lambda_1 = 1, \qquad \lambda_2 = \frac{1-\theta}{\theta} \tag{47}$$

- **Below a Half:** The second root exceeds one, so $\lambda = 1$ is used.
- **Above a Half:** The second root falls below one and is used instead.

> **A Unit Root Is Non-Explosive but Not Stationary**
> In the first case the root used equals one exactly. Every shock to inflation is then permanent.

What Plays the Root's Role in a System?

One equation had one root to check against the unit circle.

> **Question**
> A model of output, inflation and the interest rate has three equations. What do you check in place of a single root?

Take answers from the room.

SECTION 3

Solving the System

Systems of Rational Expectations Equations

Stack the variables into a vector. The first-order equation is then a system:

$$Z_t = B\, E_t Z_{t+1} + X_t \tag{48}$$

where:

- Z_t: endogenous variables
- B: matrix of forward weights
- X_t: driving variables
- n: number of variables

> **This Z_t Differs from the Data Vector of Lecture 1.4**
> Here Z_t stacks the model's endogenous variables. In Lecture 1.4 it stacked observed data.

The Scalar Forward Solution Carries over to the System

Set the system under the scalar equation, and mark what changes:

$$
\begin{aligned}
y_t &= x_t + a\, E_t\, y_{t+1} &\implies&& y_t &= \sum_{k=0}^{\infty} a^k E_t\, x_{t+k} \\
Z_t &= X_t + B\, E_t\, Z_{t+1} &\implies&& Z_t &= \sum_{k=0}^{\infty} B^k E_t\, X_{t+k}
\end{aligned}
\tag{49}
$$

The first section wrote out every scalar substitution; the link returns there.

> **Each Substitution Multiplies by the Matrix Again**
> Repeated substitution runs as before. The remainder is $B^N E_t Z_{t+N}$, which must vanish.

REPEATED SUBSTITUTION

Eigenvalues of the Coefficient Matrix

The sum converges only under a matrix version of $|a| < 1$. Along each eigenvector, B just rescales by its eigenvalue:

$$Be_i = \lambda_i e_i \qquad \Longrightarrow \qquad BV = V\Lambda \tag{50}$$

where:

- λ_i: eigenvalue of B
- e_i: its eigenvector
- V: eigenvector matrix
- Λ: diagonal eigenvalue matrix

The System Splits into Scalar Equations

Lecture 1.2 met eigenvalues. With n distinct ones, each eigenvector direction has its own scalar root, λ_i. The eigenvector matrix is then invertible, so $B = V\Lambda V^{-1}$.

Why High Powers Depend Only on the Eigenvalues

Squaring $B = V\Lambda V^{-1}$ puts an inverse pair in the middle, and the pair cancels:

$$B^2 = V\Lambda \underbrace{V^{-1}V}_{\text{identity}} \Lambda V^{-1} = V\Lambda^2 V^{-1} \tag{51}$$

Multiplying by B once more adds one more pair, which also cancels:

$$B^3 = V\Lambda^2 \underbrace{V^{-1}V}_{\text{identity}} \Lambda V^{-1} = V\Lambda^3 V^{-1} \tag{52}$$

The Stability Condition for a Forward-Looking System

After n products, $n-1$ pairs have cancelled. Only the eigenvalues are raised to the power:

$$B^n = V\Lambda^n V^{-1}, \qquad \Lambda^n = \text{diag}(\lambda_1^n, \ldots, \lambda_n^n) \tag{53}$$

Each eigenvalue's modulus, not its sign, settles whether its direction shrinks or explodes:

- **Inside the Unit Circle:** $|\lambda_i| < 1$, so λ_i^n shrinks to zero.
- **Outside the Unit Circle:** $|\lambda_i| > 1$, so $|\lambda_i^n|$ grows without bound.

> **Every Eigenvalue of B Inside Ensures a Stable Solution**
> With every $|\lambda_i| < 1$, $B^n \to 0$ and the forward solution is unique and stable. Here every variable is a jump variable.

Simulated Paths Under Stable, Unit and Explosive Roots

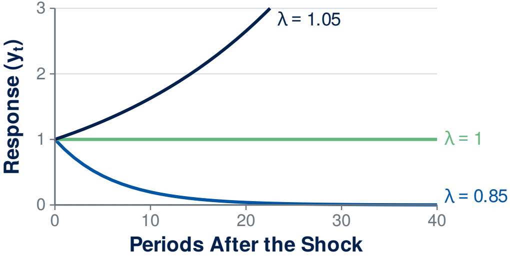

Figure 1: Response of $y_t = \lambda y_{t-1} + \epsilon_t$ to one unit shock at $t = 0$, for three roots.

> **Only Inside the Unit Circle Does a Shock Die Away**
> At 0.85, under 4 per cent of the shock remains after twenty periods. At 1, all of it remains.

Counting Explosive Roots Against Jump Variables

Capital cannot jump and a share price can. A condition requiring every eigenvalue inside the unit circle is then too strong:

- **Predetermined Variables:** Fixed by the past at t, such as capital.
- **Jump Variables:** Free to reset at t, such as a price.

> **Each Jump Variable Needs Exactly One Explosive Root**
> A unique stable path needs one explosive root per jump variable (Blanchard & Kahn, 1980). Too few gives many stable paths; too many gives none.

Calculating the Eigenvalues of a Small System

A two-by-two matrix (A) has its eigenvalues where $A - \lambda I$ is singular:

$$\det(A - \lambda I) = 0 \qquad \Longrightarrow \qquad \lambda^2 - \mathrm{tr}(A)\,\lambda + \det(A) = 0 \tag{54}$$

The trace is the two roots' sum, and the determinant their product:

$$\lambda_1 + \lambda_2 = a_{11} + a_{22}, \qquad \lambda_1\lambda_2 = a_{11}a_{22} - a_{12}a_{21} \tag{55}$$

where:

- I: identity matrix

The Generalised Schur Form of a Linear Model

Separating stable from explosive directions starts from this form (Klein, 2000):

$$\Gamma E_t Z_{t+1} = M Z_t + \Psi X_t \tag{56}$$

where:

- Γ: matrix on expectations
- M: matrix on today's variables
- Ψ: loading on driving variables

> **Γ Need Not Have an Inverse**
> A static equation gives Γ a row of zeros. The generalised Schur method still works.

Generalised Schur Finds the Stable Path in Three Steps

The method triangularises both matrices at once, then sorts and counts the roots:

1. **Triangularise:** Find the unitary matrices (Q, Z) that make $Q\Gamma Z$ and QMZ triangular.
2. **Sort:** Order the generalised eigenvalues so the stable ones come first.
3. **Count:** Compare the explosive roots against the jump variables.

> **The Matrix Z Is Not the Vector Z_t**
> Z is one of the two unitary matrices. Z_t is still the vector of endogenous variables.

Reading the Solution Off the Triangular Blocks

Ordering puts the explosive block last, and the terminal condition zeroes it:

$$u_t = \underbrace{Z_{21} Z_{11}^{-1}}_{\text{the policy function}} k_t \tag{57}$$

where:

- k_t: predetermined and driving variables
- u_t: jump variables
- Z_{11}, Z_{21}: blocks of Z

> **The Count Makes Z_{11} Square**
> The formula also needs Z_{11} invertible (Klein, 2000).

Binder and Pesaran Remove the Lag with a Matrix C

Binder and Pesaran (1996) swap λ for a lag-removing matrix (C):

$$Z_t = AZ_{t-1} + B\,E_tZ_{t+1} + HX_t, \qquad W_t \equiv Z_t - CZ_{t-1} \tag{58}$$

where:

- A: lag matrix
- H: loading on driving variables
- W_t: constructed vector

C Solves a Matrix Quadratic

Repeating the scalar substitutions with matrices leaves one bracket on Z_{t-1}:

$$(I - BC)\, W_t = B\, E_t W_{t+1} + \left(BC^2 - C + A\right) Z_{t-1} + HX_t \tag{59}$$

Iteration Is One Way to Find C

Solving $BC^2 - C + A = 0$ is non-trivial. One method iterates $C_n = BC_{n-1}^2 + A$ from $C_0 = I$.

From the Solved System to a Simulable Model

The solved system keeps the scalar solution's lagged term and forward sum. A VAR for the driving variables collapses the sum:

$$X_t = \Phi X_{t-1} + \epsilon_t \qquad \Longrightarrow \qquad Z_t = CZ_{t-1} + \Pi X_t \tag{60}$$

where:

- Φ: VAR matrix of X_t
- Π: loading on X_t

A Solved Model Is Two Equations a Computer Can Run

Stacking the two equations gives one VAR in the endogenous and driving variables. Only the shocks ϵ_t enter from outside.

Putting Any Linear Model into the Solver

Several variables dated t can share an equation. The solver starts from this form:

$$KZ_t = AZ_{t-1} + B\,E_t Z_{t+1} + HX_t \tag{61}$$

where:

- K: contemporaneous matrix

> **Four Matrices Specify Any Linear Model**
> The modeller supplies K, A, B and H, mostly zeros. The solver multiplies through by K^{-1} and returns C and Π.

SECTION 4

The Lucas Critique

What Changes When the Central Bank Changes Its Rule?

The policy rule is a structural equation, with entries in the four matrices.

> **Question**
> A central bank adopts a new interest-rate rule. Does a VAR estimated on the last decade still describe the economy?

Take answers from the room.

Permanent Income as a Structural Equation

Consumption follows expectations of after-tax income (y_t), in the forward-solution form:

$$c_t = \gamma \sum_{k=0}^{\infty} \beta^k E_t y_{t+k} \tag{62}$$

where:

- c_t: consumption
- γ: propensity to consume
- β: discount factor

> **After-Tax Income Reuses a Familiar Symbol**
> Until now y_t was a generic endogenous variable. In this section it is after-tax income.

Income Growing at a Constant Rate Is an AR(1)

Let income grow at a constant expected growth rate (g). Set it under section 2's AR(1):

$$\begin{aligned} x_t &= \rho \, x_{t-1} + \epsilon_t \\ y_t &= (1+g) \, y_{t-1} + \epsilon_t \end{aligned} \tag{63}$$

So the AR(1) forecast holds, with $1+g$ in place of ρ:

$$E_t \, y_{t+k} = (1+g)^k \, y_t \tag{64}$$

The Reduced Form Depends on the Income Process

Section 2's multiplier sums the forecasts. Here β plays α, and $1 + g$ plays ρ:

$$\sum_{k=0}^{\infty} \alpha^k E_t\, x_{t+k} = \frac{x_t}{1 - \alpha\,\rho}$$
$$\sum_{k=0}^{\infty} \beta^k E_t\, y_{t+k} = \frac{y_t}{1 - \beta\,(1+g)} \tag{65}$$

Multiply by γ. With $\beta(1 + g) < 1$, consumption is:

$$c_t = \frac{\gamma}{1 - \beta(1+g)}\, y_t \tag{66}$$

The Lucas Critique of Econometric Models

The structural equation holds whatever the income process; the reduced form does not:

- **What Is Structural:** The propensity γ and the discount factor β.
- **What Is Not:** The estimated coefficient, which holds the income process as well.
- **The Consequence:** A policy changing expectations changes the estimated coefficient.

> **Deep Parameters Survive a Change of Policy**
> Deep parameters describe preferences and technology. Lucas argued these stay fixed when policy changes; reduced-form coefficients need not (Lucas, 1976).

A Temporary Tax Cut: What the Advisers Estimate

A government plans a one-period income tax cut. Its advisers regress consumption on after-tax income, using years of data:

- **The Advisers:** Each euro of cut raises consumption by the regression coefficient.
- **The Household:** Consumes only γ of each euro; the rise lasts one period.

> **The Regression Overstates a One-Period Rise**
> The estimate was built on income that kept growing. A one-period rise is not permanent growth.

The Advisers' Estimate Is About 32 Times Too Large

The household's true response is γ. At Whelan's discount factor of 0.95 and growth of two per cent:

$$\frac{\gamma}{1-\beta(1+g)} = \frac{\gamma}{1-0.95\times 1.02} = \frac{\gamma}{1-0.969} = \frac{\gamma}{0.031} \approx 32\gamma \tag{67}$$

More Data Would Not Correct the Error

The advisers estimate the wrong object for this question. More data only sharpens that estimate.

The Critique Reaches Every Policy Rule

Monetary, fiscal and exchange-rate rules all shape a DSGE model's reduced form:

- **Reduced-Form Coefficients:** Each can change when the rule changes.
- **The VAR's Limit:** Fixed coefficients cannot represent a change in the policy rule.

> **A DSGE Model Derives the VAR from the Rule**
> Change the rule and the derived VAR changes. Whelan calls this a selling point for DSGE models.

Which Estimates Would Survive a Regime Change?

A VAR's coefficients are estimated under the sample's policy rule.

> **Question**
> Take the identified monetary policy shock from Lecture 1.3. Which parts of that exercise does the Lucas critique threaten, and which survive?

Take answers from the room.

What the Front Half of the Module Built

Five lectures built the evidence and the machinery. The class test covers both:

- **The Evidence:** Detrending, VARs, identification, impulse responses.
- **The Machinery:** State space, the Kalman filter, and today's solution methods.

> **Reduced-Form Evidence Is Still the Test**
> Estimated responses hold under the sample's policy rule. A model must match them before it is asked about another rule.

Before Lecture 2.1

1. **Play:**
   Simulate $y_t = \lambda y_{t-1} + \epsilon_t$ in R, at $\lambda = 0.85$, 1 and 1.05.

2. **Apply:**
   Revise Lectures 1.1 to 1.5 for the class test.

3. **Read:**
   Sections 5.3 to 5.5 of Romer (2019), before Lecture 2.1.

Next Lecture

Lecture 2.1 builds the first model with optimising households:

1. **The Planner's Problem.**
   One planner, one technology shock, no frictions.
2. **The First-Order Conditions.**
   From a Lagrangian to six equations.
3. **Log-Linearising.**
   Around a non-stochastic steady state.
4. **What the RBC Model Gets Wrong.**
   Money, propagation, and hours worked.

Thank you

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BACKUP

Appendix

Notation

Symbol	What it is	First met
y_t	endogenous variable	The First-Order Stochastic Difference Equation
x_t	driving variable	The First-Order Stochastic Difference Equation
a	forward weight	The First-Order Stochastic Difference Equation
E_t	time-t expectation	The First-Order Stochastic Difference Equation
N	number of substitutions	Repeating the Substitution to the Limit
k	periods ahead	Repeating the Substitution to the Limit
P_t	asset price	Pricing an Asset With No Risk Premium
D_t	dividend	Pricing an Asset With No Risk Premium
r	safe return	Pricing an Asset With No Risk Premium
ϵ_t	unpredictable forecast error	The Same Equation Solved Backwards
ρ	persistence of x	From Structural Equation to Reduced Form
ϵ_t	fresh shocks	From Structural Equation to Reduced Form
b	forward weight	The Second-Order Stochastic Difference Equation
a	backward weight	The Second-Order Stochastic Difference Equation

Notation

Symbol	What it is	First met
v_t	constructed variable	A Change of Variable That Removes the Lag
λ	unknown coefficient	A Change of Variable That Removes the Lag
α	placeholder weights	A Change of Variable That Removes the Lag
β	placeholder weights	A Change of Variable That Removes the Lag
λ_1	characteristic roots	The Other Root Sets the Forward Discount Factor
λ_2	characteristic roots	The Other Root Sets the Forward Discount Factor
π_t	inflation rate	A Hybrid New Keynesian Phillips Curve
γ_f	forward weight	A Hybrid New Keynesian Phillips Curve
γ_b	backward weight	A Hybrid New Keynesian Phillips Curve
κ	slope on inflationary pressure	A Hybrid New Keynesian Phillips Curve
θ	forward weight	The Hybrid Curve Is a Second-Order Equation
Z_t	endogenous variables	Systems of Rational Expectations Equations
B	matrix of forward weights	Systems of Rational Expectations Equations
X_t	driving variables	Systems of Rational Expectations Equations

Notation

Symbol	What it is	First met
n	number of variables	Systems of Rational Expectations Equations
λ_i	eigenvalue of B	Eigenvalues of the Coefficient Matrix
e_i	its eigenvector	Eigenvalues of the Coefficient Matrix
V	eigenvector matrix	Eigenvalues of the Coefficient Matrix
Λ	diagonal eigenvalue matrix	Eigenvalues of the Coefficient Matrix
I	identity matrix	Calculating the Eigenvalues of a Small System
A	two-by-two matrix	Calculating the Eigenvalues of a Small System
Γ	matrix on expectations	The Generalised Schur Form of a Linear Model
M	matrix on today's variables	The Generalised Schur Form of a Linear Model
Ψ	loading on driving variables	The Generalised Schur Form of a Linear Model
Q	unitary matrices	Generalised Schur Finds the Stable Path in Three Steps
Z	unitary matrices	Generalised Schur Finds the Stable Path in Three Steps
k_t	predetermined and driving variables	Reading the Solution Off the Triangular Blocks
u_t	jump variables	Reading the Solution Off the Triangular Blocks

Notation

Symbol	What it is	First met
Z_{11}, Z_{21}	blocks of Z	Reading the Solution Off the Triangular Blocks
A	lag matrix	Binder and Pesaran Remove the Lag With a Matrix C
H	loading on driving variables	Binder and Pesaran Remove the Lag With a Matrix C
W_t	constructed vector	Binder and Pesaran Remove the Lag With a Matrix C
C	lag-removing matrix	Binder and Pesaran Remove the Lag With a Matrix C
Φ	VAR matrix of X_t	From the Solved System to a Simulable Model
Π	loading on X_t	From the Solved System to a Simulable Model
K	contemporaneous matrix	Putting Any Linear Model Into the Solver
c_t	consumption	Permanent Income as a Structural Equation
γ	propensity to consume	Permanent Income as a Structural Equation
β	discount factor	Permanent Income as a Structural Equation
y_t	after-tax income	Permanent Income as a Structural Equation
g	expected growth rate	Income Growing at a Constant Rate Is an AR(1)

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