LECTURE 3.1

Growth Facts and the Solow Model

ECON42550 Macroeconomics

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Where We Are

1. Short-Run Fluctuations

1.1 The Keynesian Multiplier

1.2 Introducing IS-MP-PC

1.3 Analysing IS-MP-PC

1.4 The Zero Lower Bound

1.5 Banks, Runs, the Cycle

2. NK Building Blocks

2.1 Expectations, Investment

2.2 Consumption, Fiscal Policy

2.3 Sticky Prices and the NKPC

3. Long-Run Growth

3.1 Growth Facts, Solow

3.2 The Fundamental Causes

3.3 Endogenous Growth

3.4 Schumpeterian Growth

> **From the Cycle to the Trend**
> Not why output fluctuates around its trend, but what sets the trend.

Last Week's Lecture

Last week we derived a Phillips curve from firms that cannot reset their prices at will.

1. **Why Prices Are Sticky.**
   Establish the fact the whole of Section 1 rested on, and the theories that explain it.
2. **The Calvo Model.**
   Derive a Phillips curve from firms that cannot reset their prices at will.
3. **The New Keynesian Phillips Curve.**
   Read the result, and confront it with the data.

Lecture Plan

1. **The Facts.**
   Establish what has to be explained before building anything to explain it: growth compounds, and countries do not all converge.

2. **The Solow Model.**
   Build the benchmark model of capital accumulation, solve it, and use it on two questions the exam asks.

3. **Solow Confronted with Data.**
   Use the same production function backwards, to decompose measured growth.

> **Whelan, Chapters 10 and 11**
> Reading: Whelan (2023, chs. 10 and 11).

SECTION 1

The Facts

Why Growth Is the Only Thing That Matters in the Long Run

This section sets out what any growth model has to explain. A recession costs an economy a few per cent of output for a few years, and then it ends. A growth rate half a point lower costs it everything, because the loss never stops accruing:

- **Nothing Else in Macroeconomics Compounds:** A demand shock decays. A growth-rate difference does not: it is applied again to a bigger base every year.
- **The Long Run Dominates:** Beyond about a decade, the growth rate matters more than the cycle, however loudly the short run is discussed.

> **These Models Explain Averages, Not Spread**
> Every model in this section explains the average. None of them says anything about the distribution of income within a country.

The Arithmetic of Compounding

$$Y_T = Y_0 (1 + g_Y)^T \tag{1}$$

- Y_0: output today
- Y_T: output T years from now
- g_Y: the annual growth rate of output

Two Countries, One Point Apart
Two countries start level. Over a century **2%** a year multiplies income by about **7** and **3%** by about **19** — nearly three times richer.

Income Doubles Every $70/g$ **Years**
With g in per cent: at 2% that is 35 years, at 3% about 23.

Convergence, and Its Clubs

If capital is the story, poor countries should grow faster than rich ones: they hold less of it, so an extra machine is worth more. The data answer in two parts:

- **Across All Countries:** Since 1960 growth is close to flat against initial income, and the poorest group has not closed the gap.
- **Inside the OECD:** Across OECD members, US states and European regions, poorer starting points do grow faster.

> **Poorer Countries Should Grow Faster, Unconditionally**
> **Absolute convergence** is the claim that a country starting poorer grows faster, with no conditions attached, so that world incomes draw together.

Convergence Happens Inside a Club

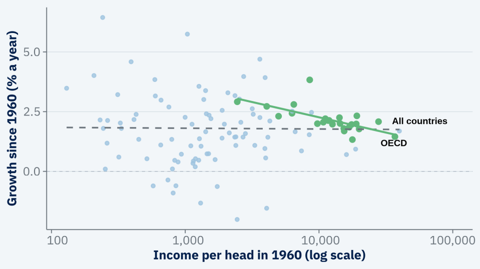

Figure 1: Average annual growth since 1960 against income in 1960: all countries, and the OECD alone.

> **OECD Slopes Down, World Stays Flat**
> **How to read it.** One point per country: 1960 income across, growth since up. Absolute convergence would be a **downward** slope.

Put the Claim in a Regression

$$\underbrace{\frac{\ln y_{iT} - \ln y_{i0}}{T}}_{\text{average annual growth}} = \beta_0 + \beta_1 \ln y_{i0} \tag{2}$$

- y_{i0}: income per worker in country i at the start
- y_{iT}: income per worker in country i T years later
- β_1: the convergence coefficient

> **β_1 Comes Out Near Zero**
> Absolute convergence requires $\beta_1 < 0$.

Hold Constant What They Are Converging To

Add controls for where each country is heading:

$$\frac{\ln y_{iT} - \ln y_{i0}}{T} = \beta_0 + \beta_1 \ln y_{i0} + \beta_3 Z_i \tag{3}$$

- Z_i: controls — saving, population growth, schooling, institutions

> **Countries Converge to Their Own Steady States**
> β_1 is near zero unconditionally and reliably **negative** once Z_i is in. Countries converge to their own steady states, not to each other's.

Growth, Poverty and Inequality

Growth matters for what it has done to how people live:

- **What Growth Has Done:** Extreme poverty fell fastest after 1990.
- **What Growth Has Not Done:** Within-country inequality rose in most rich economies.

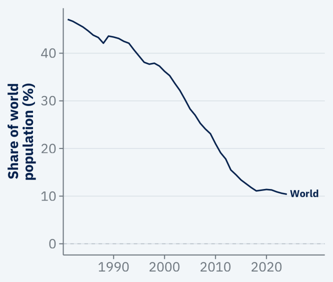

Figure 2: The share of the world's population below $3.00 a day, 2021 PPP.

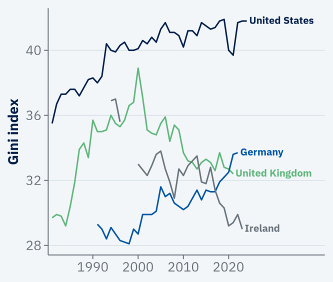

Figure 3: Gini indices for four advanced economies over the same period.

Absolute Convergence, or Conditional?

The regressions gave two different answers about poor countries catching up.

> **Question**
> What is the difference between absolute and conditional convergence, and which one do the cross-country data support?

Take answers from the room.

SECTION 2

The Solow Model

Households and Firms

This section builds the benchmark model of capital accumulation. Neither of its two agents optimises:

- **Households:** Own capital, supply labour inelastically, save a fixed share s of income.
- **Firms:** Competitive, identical, hiring labour and renting capital under one technology.

> **Why a Fixed Saving Rate Is a Feature**
> Everything the model predicts follows from accumulation and diminishing returns, not from preferences. A clever household would hide which assumption does the work.

The Production Function

Capital (K_t), workers (L_t) and technology (A_t) combine to produce output:

$$Y_t = K_t^{\alpha} \big(\underbrace{A_t L_t}_{\text{effective labour}} \big)^{1-\alpha} \tag{4}$$

- α: the capital share, $0 < \alpha < 1$

> **α Is a Different Parameter Here**
> In Section 1, α measured how far demand falls when the real rate rises above r^*. Here it is the **capital share of output**, about 1/3.

> **Technology Multiplies Labour, Not Output**
> Only labour-augmenting technology gives a balanced growth path; under Cobb–Douglas the placement makes no difference.

Factor Prices, and Where the Capital Share Comes From

Competitive firms pay each factor its marginal product. The real return to owning capital is the rental rate net of depreciation ($R_t - \delta$):

$$w_t = (1-\alpha)\frac{Y_t}{L_t}, \qquad R_t = \alpha\frac{Y_t}{K_t} \tag{5}$$

- w_t: the wage per worker
- R_t: the rental rate on a unit of capital

Multiply out and the two payments exhaust output exactly:

$$Y_t = w_t L_t + R_t K_t \tag{6}$$

The Labour Exponent Is the Labour Share

Rearranged, $1 - \alpha = w_t L_t / Y_t$: the exponent on labour **is** the labour share of national income, which is why α is pinned at 1/3 rather than estimated.

Output per Effective Worker

$$\tilde{y}_t = \tilde{k}_t^{\alpha}, \qquad \tilde{k}_t \equiv \frac{K_t}{A_t L_t}, \qquad \tilde{y}_t \equiv \frac{Y_t}{A_t L_t} \tag{7}$$

A Tilde Means per Effective Worker

Lower case now means **per effective worker**, not a log; K_t/L_t itself keeps growing for ever.

The Fundamental Law of Motion

$$K_{t+1} = K_t + \underbrace{sY_t - \delta K_t}_{\text{net investment}} \tag{8}$$

The law of motion is the model's only dynamic equation; everything else is an identity or a first-order condition.

Saving Is Investment, by Assumption
The economy is closed and has no government, so every euro saved buys new capital.

The Law of Motion, per Effective Worker

Divide $K_{t+1} = (1 - \delta)K_t + sY_t$ by $A_{t+1}L_{t+1}$:

$$(1+n)(1+g)\,\tilde{k}_{t+1} = s\,\tilde{k}_t^{\alpha} + (1-\delta)\,\tilde{k}_t \qquad \textbf{(9)}$$

For small labour-force growth (n) and technology growth (g) the cross-product ng is negligible, and capital per effective worker changes by:

$$\Delta\tilde{k}_{t+1} = s\,\tilde{k}_t^{\alpha} - \underbrace{(n+g+\delta)\,\tilde{k}_t}_{\text{break-even investment}} \qquad \textbf{(10)}$$

$\tilde{k}$ Rises Only When Saving Beats Break-Even

$(n+g+\delta)\tilde{k}$ replaces wear, equips new workers and keeps up with technology.

Capital Stops Growing Where Saving Meets Break-Even

Set $\Delta \tilde{k}_{t+1} = 0$ and solve for the **steady state** $\tilde{k}^*$:

$$s\,\tilde{k}^{*\alpha} = (n + g + \delta)\,\tilde{k}^* \tag{11}$$

$$\tilde{k}^* = \left(\frac{s}{n+g+\delta}\right)^{\frac{1}{1-\alpha}}, \qquad \tilde{y}^* = \left(\frac{s}{n+g+\delta}\right)^{\frac{\alpha}{1-\alpha}} \tag{12}$$

The Star Is Still a Resting Point, Not an Optimum
As in Section 1, nobody maximised anything to get $\tilde{k}^*$; the golden rule later does.

The Steady State, Drawn

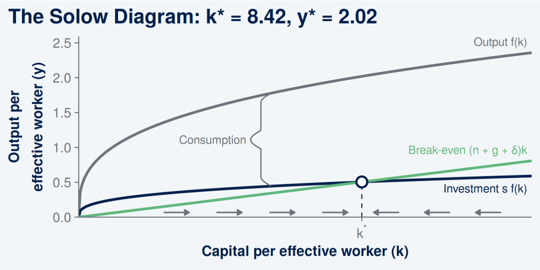

Figure 4: Output per effective worker (dashed), saving beneath it, and the straight break-even line, which cuts the saving curve once at $\tilde{k}^*$. The gap between output and saving is shaded as consumption, and the range over which capital is still rising is shaded too.

> **Capital Grows Below $\tilde{k}^*$, Shrinks Above**
> **How to read it.** The curve is saving $s\tilde{k}^\alpha$, the line is break-even $(n + g + \delta)\tilde{k}$. Diminishing returns flatten the saving curve until break-even overtakes it.

You Always Arrive, and You Never Overshoot

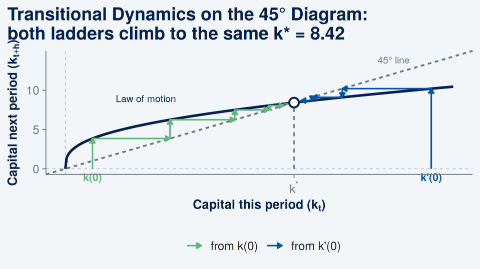

Figure 5: The 45-degree diagram with the law of motion, and two staircases: one climbing from a starting capital stock below $\tilde{k}^*$, one stepping down from one above it. Both land on the same $\tilde{k}^*$.

> **Transitional Dynamics: Every Step Gets Smaller**
> **How to read it.** Each step toward $\tilde{k}^*$ is smaller, as the gap between the lines closes.

Comparative Statics: Saving and Depreciation

The model produces conditional convergence: two countries with different s or n head for different steady states, which is what β_1 needed Z_i to see:

- **A Higher Saving Rate:** Raises $\tilde{k}^*$ and $\tilde{y}^*$ permanently.
- **Faster Depreciation or Population Growth:** A steeper break-even line lowers $\tilde{k}^*$ and $\tilde{y}^*$.

Saving Raises the Level, Not Growth

A higher saving rate raises the **level** of income for ever and the **growth rate** only in transition. No saving rate produces permanent growth in $\tilde{y}$.

Comparative Statics: Technology, and What Saving Cannot Do

In steady state $\tilde{y}$ is constant, so $Y_t/L_t = A_t\tilde{y}^*$ grows at g, **the growth rate of technology, handed to the model from outside it.** The model does not explain growth; it explains why capital accumulation cannot be the explanation:

- **Saving:** Buys a one-off, permanent rise in the level of income.
- **Technology:** Raises income for ever, from a source the model never names.

> **The Rest of Section 3 Explains g**
> 3.2 asks what makes some countries better at adopting technology, 3.3 makes g the outcome of research effort, and 3.4 of competition between firms.

Solow in Continuous Time

The same law of motion, with every Δ written as a dot and the exact $(1 + n)(1 + g)$ term gone rather than approximated away:

$$\dot{\tilde{k}} = s\,\tilde{k}^{\alpha} - (n + g + \delta)\,\tilde{k} \tag{13}$$

A Dot Means a Time Derivative

$\dot{x} = dx/dt$, and $\dot{x}/x$ is the **growth rate** of x. The economics is identical to discrete time; the algebra is shorter, which is why the exam uses it.

The Multiplicative Form Is the Same Model

Past papers write $Y_t = B_t K_t^{\alpha} L_t^{1-\alpha}$, with B_t multiplying everything, and get $k^* = \left(sB/(n+\delta)\right)^{1/(1-\alpha)}$ — with B constant. That is the same function as ours:

$$BK^{\alpha}L^{1-\alpha} = K^{\alpha}(AL)^{1-\alpha}, \qquad A = B^{\frac{1}{1-\alpha}} \tag{14}$$

- B_t: total factor productivity, technology written as a multiplicative shifter

> **Two Forms, Two Growth Rates**
> The multiplicative rate is $g^B = (1-\alpha)g$, so the two are **not** interchangeable. A and g stay labour-augmenting; use whichever form the question uses.

Consumption Is What Is Left After Break-Even

$$\tilde{c}^* = \tilde{k}^{*\alpha} - \underbrace{(n + g + \delta)\,\tilde{k}^*}_{\text{break-even investment}} \tag{15}$$

In the steady state, saving covers break-even investment exactly:

$$s\,\tilde{k}^{*\alpha} = (n + g + \delta)\,\tilde{k}^* \tag{16}$$

- $\tilde{c}^*$: steady-state consumption per effective worker

> **More Capital Eventually Is Not Worth It**
> Every extra unit of capital adds $\alpha\tilde{k}^{*\alpha-1}$ to output and costs $n + g + \delta$ to keep. The first falls as capital rises, the second does not.

There Is a Best Saving Rate, and It Is the Capital Share

Choose $\tilde{k}^*$ to maximise $\tilde{c}^*$, and differentiate:

$$\alpha \, \tilde{k}_{\text{gold}}^{\alpha-1} = n + g + \delta \tag{17}$$

- $\tilde{k}_{\text{gold}}$: the capital per effective worker maximising $\tilde{c}^*$

The steady-state condition replaces break-even investment on the right:

$$\alpha \, \tilde{k}_{\text{gold}}^{\alpha-1} = \boxed{s \, \tilde{k}_{\text{gold}}^{\alpha-1}} \qquad \Longrightarrow \qquad s_{\text{gold}} = \alpha \tag{18}$$

Save the Capital Share, and No More
Consumption is highest where the **marginal product of capital equals break-even investment**.

Saving Too Much Is Possible

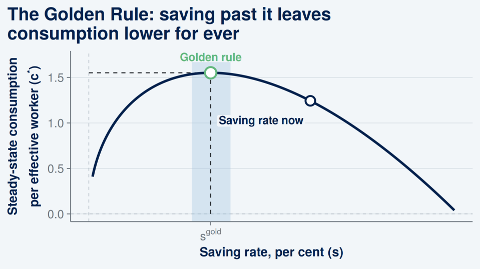

Figure 6: Steady-state consumption per effective worker plotted against the saving rate. The hump peaks at the golden rule, $s = \alpha \approx 0.33$; the economy is marked at $s = 0.60$, well down the right-hand side.

> **The Saving Rate Is on the Horizontal Axis**
>
> **How to read it.** Not capital across, but s: each point is the consumption a country would settle at if it saved that share for ever.

Past the Golden Rule, Saving Less Is a Free Lunch

A country to the right of $\tilde{k}_{\text{gold}}$ has over-accumulated capital.

Saving Less Is Sometimes a Free Lunch

Past the golden rule, saving less raises consumption **now and in every future period**. Advanced economies are not there: measured returns to capital exceed growth rates.

The One Star That Is an Optimum

$\tilde{k}_{\text{gold}}$ is the exception: every other star in this course marks a resting point, and this one solves a maximisation problem.

A Government Takes a Share of Output

Output is now split three ways, and the government takes a fixed share:

$$Y = C + I + G, \qquad G = \sigma Y \tag{19}$$

- σ: the share of output the government spends

> **The Source Paper Writes s the Other Way Up**
> SQ2 Q7 writes consumption as $(s - \lambda\sigma)Y$, so its s is the share **consumed**. Here s stays the saving rate, and the paper's s is our $1 - s$.

Government Spending Crowds Out Consumption, Saving, or Both

A euro of government spending comes out of private consumption, out of private saving, or out of both:

$$C = \left(1 - s - \lambda\sigma\right)Y \tag{20}$$

- λ: the fraction of government spending that displaces private consumption

> **λ Decides What Spending Crowds Out**
> $\lambda = 1$ means a euro of government spending crowds out a euro of private consumption; $\lambda = 0$ means it comes entirely out of private saving.

Government Spending and the Investment Share

Private investment is what remains after consumption and government spending:

$$I_{\text{priv}} = Y - C - G = \big(s - (1 - \lambda)\sigma\big)\,Y \tag{21}$$

A fraction φ of that spending is investment, so add $\varphi\sigma Y$ back:

$$I = \underbrace{\big(s - (1 - \lambda)\sigma + \sigma\varphi\big)}_{\text{the investment share}}\,Y \tag{22}$$

When Does Government Spending Raise the Steady State?

Put the new investment share into the steady-state condition:

$$\tilde{k}^* = \left(\frac{s - (1 - \lambda)\sigma + \sigma\varphi}{n + g + \delta} \right)^{\frac{1}{1-\alpha}} \tag{23}$$

Differentiate the bracket in σ. The steady state rises if and only if

$$\varphi > 1 - \lambda \tag{24}$$

At $\varphi = 0$, $\tilde{k}^*$ is unchanged when $\lambda = 1$ and falls fastest when $\lambda = 0$.

A State Helps Only If It Builds

A state that builds and taxes consumption raises $\tilde{k}^*$; a state that transfers and taxes saving lowers it.

Does a Higher Saving Rate Raise Growth For Ever?

The comparative statics gave a rise in s one permanent effect and one temporary one.

> **Question**
> What does a permanent rise in the saving rate do to the **level** of income, and what does it do to the **growth rate**?

Take answers from the room, then raise s in the Solow app and watch the growth rate.

SECTION 3

Solow Confronted With Data

Growth Accounting

This section runs the same production function backwards, to decompose measured growth. Logs turn the multiplicative form $Y_t = B_t K_t^{\alpha} L_t^{1-\alpha}$ into a sum, and differentiating it in time gives:

$$g_t^Y = g_t^B + \alpha\, g_t^K + (1-\alpha)\, g_t^L \tag{25}$$

- g_t^B: the growth rate of total factor productivity

> **g^X Is the Growth Rate of X**
> A superscript names the variable, a subscript the year: g_{2015}^K is how fast the capital stock grew in 2015, not capital raised to a power.

What the Residual Is

$$\underbrace{g_t^Y - \alpha\, g_t^K - (1-\alpha)\, g_t^L}_{\text{the Solow residual}} = g_t^B \qquad (26)$$

- α is not estimated: the national accounts give the capital share, about 1/3.
- Quality-adjusted labour puts $q_t L_t$ in place of L_t, which shrinks the residual.

> **Total Factor Productivity Is a Name, Not a Measurement**
> The residual is **output growth the measured inputs do not account for**: technical progress, better management, reallocation, utilisation, mismeasurement and accounting error.

The Productivity Slowdown

Productivity growth in the advanced economies was fast until the early 1970s and slow after. No explanation covers both.

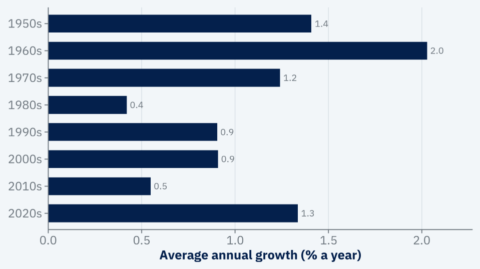

Figure 7: Multifactor productivity growth in the United States, by decade, since 1950.

> **The Pattern Holds as Levels Shift**
> Each bar's level depends on the assumed capital share; the pattern across bars does not.

The Productivity Gap Between the United States and Europe

France closed on the United States until the mid-1990s. Neither France nor the euro area has kept pace since.

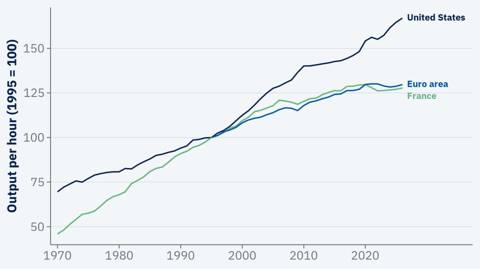

Figure 8: Real output per hour, 1995 = 100: France from 1970, the euro-area aggregate from 1995.

> **France Caught Up, Then Stopped**
> France rises from 46 to 100 between 1970 and 1995. Since then the United States reaches 167 and France 128.

Growth From Inputs Alone

Put in more capital and more workers every year and you get more output every year, and none of it is progress. The identity tells the two apart:

- Soviet growth ran three decades on capital accumulation and workers leaving the land.
- Growth accounting puts nearly all Soviet growth in inputs, nearly none in the residual.
- Diminishing returns then bite: the same investment buys less, and growth falls back.

The US Residual Dwarfs the Soviet Union's

Solow's 1957 calculation put **87.5%** of US output per man-hour growth in the residual.

No Residual, No New Growth Rate

Krugman applied this calculation to East Asia before the 1997 crisis: fast growth with no residual is a level effect working itself out, not a new growth rate.

Ireland’s Accounts

Lecture 0’s second puzzle. Same identity, same country, same years, same capital share — and two answers.

5.2%
annual growth of Irish **GDP**, 1999–2024

43%
of it claimed by the residual

2.9%
annual growth of **GNI***, over the same years

> **One Output Choice, Two Different Stories**
> On GDP the residual is **2.2 points** of the 5.2; on GNI* it is **$-$0.1** of the 2.9. Only the output series changes — and that is the problem: capital is still total-economy, so the GNI* residual is biased down.

One Slider, Two Countries

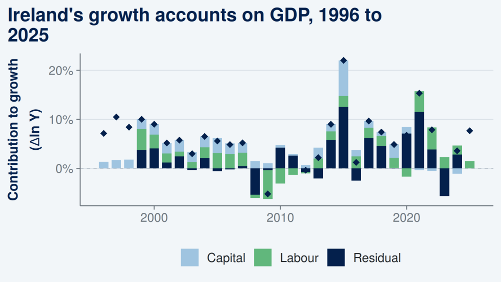

Figure 9: On GDP: each year from 1996 to 2025 split into capital, labour and the residual, with a diamond at total growth. The 2015 column is enormous.

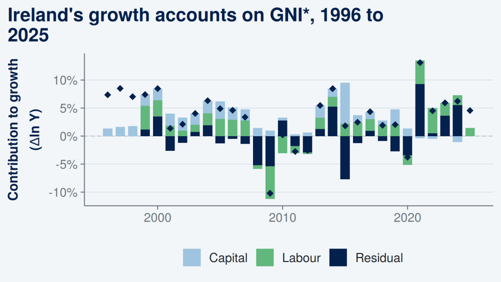

Figure 10: The same decomposition on GNI*. The 2015 column is modest, and the rest of the chart is close to flat.

> **One Year Is the Whole Argument**
>
> **How to read it.** Same identity, same years, same capital share on both panels. Look at 2015 and then at everything either side of it.

The Capital Share Does Not Sit Still

The identity cannot tell a relocated balance sheet from a new machine: in 2015 measured GDP grew **24.6%** and the net capital stock **24.7%**. After the output series, α is the second measurement choice, and it carries the decomposition:

- CSO capital share of gross value added: **0.40** in 2000, **0.60** in 2015, **0.67** in 2019.
- We set α at **one third**; Ireland's share rose nearly thirty points in two decades.

> **α Alone Sets Every Bar's Height**
> Every contribution in the decomposition is $\alpha\, g^K$ or $(1-\alpha)\, g^L$. Change α and every bar changes height, with no data series revised.

Two Economies, One Statistic

For 2015 the identity says productivity grew **12.5%**; the CSO's multifactor measure says it fell about **16%**. Split that measure in two:

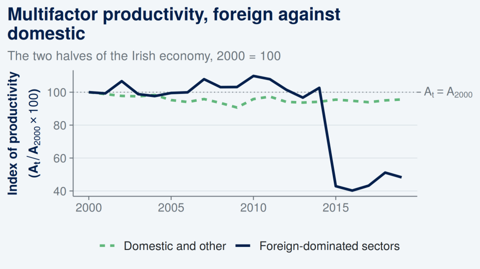

Figure 11: CSO multifactor productivity, 2000 = 100: foreign-dominated sectors against domestic.

> **The National Average Describes No One**
> Averaging the foreign and domestic indices describes nobody's experience. The identity is not wrong; somebody has to choose the series.

Capital, and What Is Left Over

One model, one decomposition that uses it, and one country where the two come apart:

1. **The Facts.**
   Growth compounds; countries converge only to their own steady states.
2. **The Solow Model.**
   Saving raises the level of output, not its long-run growth rate.
3. **Solow Confronted With Data.**
   Growth splits into accumulation and a residual Ireland's output series sets.

> **Three Attempts to Reduce Our Ignorance**
> The residual measures our ignorance. Section 3 makes three attempts to reduce it: institutions in 3.2, research in 3.3, competition in 3.4.

Before Lecture 3.2

1. **Play:**
   The Solow app. Set s to the golden-rule value and then above it, and watch consumption fall.
2. **Apply:**
   The app's government-spending exercise, stage 6. Vary σ with φ above and below $1 - \lambda$, and sign the effect each time.
3. **Review:**
   Whelan, chapter 14, on institutions.

> **No Problem Set Is Issued Today**
> Problem set 5 was due before today. Set 6 is issued at the end of Lecture 3.2 and covers 3.1 and 3.2, so today's government-spending derivation returns then.

Next Week

Today measured growth and found a residual that the model does not explain.

3.2 Asks What Sets the Residual

Lecture 3.2 separates what accounts for growth from what causes it, and makes the case for institutions.

Thank you

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BACKUP

Appendix

Notes and Tools

- **Whelan's lecture notes** — the notes for this module. Chapters 10 and 11 go with today.
- **The Solow app** — the 45° diagram, the staircase from both sides, the golden rule at stage 4 and government spending at stage 6.
- **The growth-accounting app** — the identity, and Ireland's accounts on GDP and on GNI*.

> **Whelan Is the Only Required Reading**
> Chapters 10 and 11 are the examinable text; the two apps and the notes above are optional.

References

Whelan, K. (2023). *Lecture notes on macroeconomics*. University College Dublin. https://www.karlwhelan.com/Macro2/Whelan-Lecture-Notes.pdf